Volume: 2 Issue: 8 Pages: 59-67

A Study on Conformal Mapping and Its Application

*[1] Ramu ST

*[1] Assistant Professor, Department of Mathematics, Government First Grade College for Women, Holenarasipura, Hassan, Karnataka, India.

Article Info.
E-ISSN: 2583-6528
Impact Factor (SJIF): 5.231
Peer Reviewed Journal
Available online:
www.alladvancejournal.com

Received: 21/July/2023
Accepted: 25/Aug/2023

***Corresponding Author**
Ramu ST
Assistant Professor, Department of Mathematics, Government First Grade College for Women, Holenarasipura, Hassan, Karnataka, India.

Abstract
Conformal mapping is an important technique used in complex analysis. Conformal mapping has many applications in different physical situations. If the function is harmonic equation and Laplace equation $\nabla^2 f = 0$, then the transformation of the functions through conformal mapping is also harmonic. In this topic there is brief introduction to conformal mappings and with the help of this theory and some of its applications in physical problems.

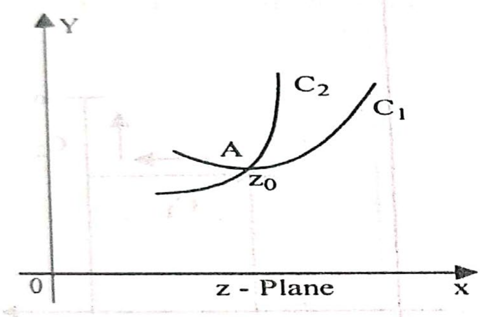
Keywords: Conformal Mapping, Connected Domain, Line Segment, Applications, Dimensions, Jacobian Transformation.

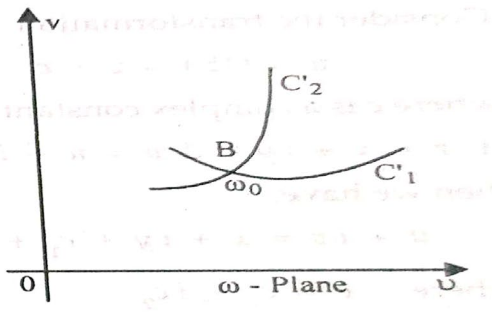
Introduction

Conformal Mapping is a function which preserves orientation and angles. that is a transformation $w = f(z)$ is said to be conformal if it preserves angle between oriented curves in magnitude as well as orientation, i.e., direction

Example

Let c_1 and c_2 be two continuous curves in the z-plane intersecting at the point z_o. Further let the curves c_1 and c_2 be transformed to the curves c_1^1 and c_2^1 under the transformation w=f(z). Let c_1^1 and c_2^1 be intersect at w_o on the w-plane where w_o= f(z_o).





$$w = f(z) = z$$

Let $w = Re^{i\phi}$, $z = re^{i\theta}$

$$\therefore w = z$$

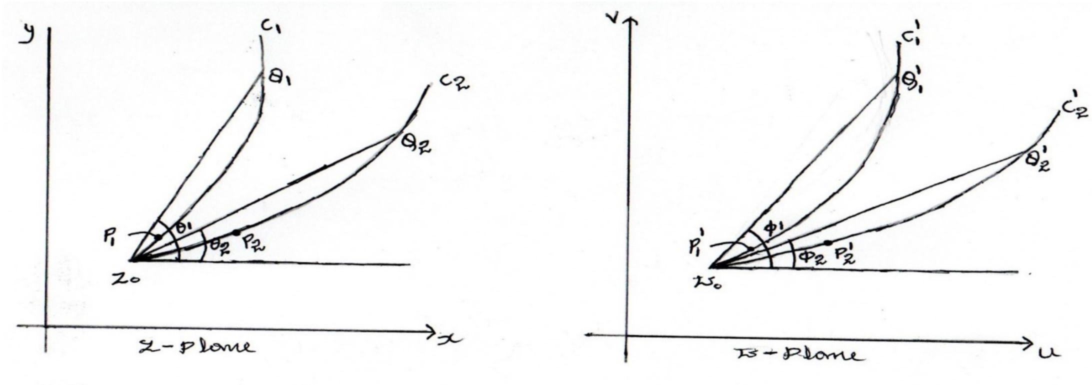
i.e. $Re^{i\phi} = re^{i\theta}$

i.e. $R = r$ $\phi = \theta$

$\phi = \theta \Rightarrow w = z$ is conformal

Theorem

If $f(z)$ is analytic in a domain D and $f'(z) \neq 0$, then transformation $w = f(z)$ is Conformal.



Proof: Let $w = f(z)$ be analytic and $f'(z) \neq 0$, Let c_1 be a curve and $P_1(z)$ and $Q_1(z + \delta z)$ be two points on the curve c_1 in the $z -$ Plane.

Let $\delta z = r_1 e^{i\theta_1}$

Let c_1^1 be a curve and $P_1^1(w)$ and $Q_1^1(w + \delta w)$ be two points on the curve c_1^1 in the $w -$ Plane.

Let $\delta w = R_1 e^{i\phi_1}$

$f: c_1 \longrightarrow c_1^1$ (i.e. Every points on the curve c_1 onto the points on the curve c_1^1)

$$\frac{\delta w}{\delta z} = \frac{R_1 e^{i\phi_1}}{r_1 e^{i\theta_1}} = \frac{R_1}{r_1} e^{i(\phi_1 - \theta_1)}$$

Taking limit as $\delta z \to 0$, we get

$$f'(z) = \lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \lim_{\delta z \to 0} \frac{R_1}{r_1} \cdot \lim_{\delta z \to 0} e^{i(\phi_1 - \theta_1)}$$

$$f'(z) = Re^{i\phi} \neq 0 \ (\because f'(z) \neq 0)$$

$$Re^{i\phi} = \lim_{\delta z \to 0} \frac{R_1}{r_1} \cdot \lim_{\delta z \to 0} e^{i(\phi_1 - \theta_1)}$$

$$Re^{i\phi} = k_1 e^{i(\phi_1 - \theta_1)}$$

$$R = k_1, \phi = \phi_1 - \theta_1 \ \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let c_2 be a curve and $P_2(z)$ and $Q_2(z + \delta z)$ be two points on the curve c in the $z -$ Plane.

Let $\delta z = r_2 e^{i\theta_2}$

Let c_2^1 be a curve and $P_2^1(w)$ and $Q_2^1(w + \delta w)$ be two points on the curve c_2^1 in the $w -$ Plane.

Let $\delta w = R_2 e^{i\phi_2}$

$f: c_2 \longrightarrow c_2^1$ (i.e. Every points on the curve c_2 onto the points on the curve c_2^1)

$$\frac{\delta w}{\delta z} = \frac{R_2 e^{i\phi_2}}{r_2 e^{i\theta_2}} = \frac{R_2}{r_2} e^{i(\phi_2 - \theta_2)}$$

Taking limit as $\delta z \to 0$, we get

$$f'(z) = \lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \lim_{\delta z \to 0} \frac{R_2}{r_2} . \lim_{\delta z \to 0} e^{i(\phi_2 - \theta_2)}$$

$$f'(z) = Re^{i\phi} \neq 0 \ (\because \ f'(z) \neq 0 \ )$$

$$Re^{i\phi} = \lim_{\delta z \to 0} \frac{R_2}{r_2} . \lim_{\delta z \to 0} e^{i(\phi_2 - \theta_2)}$$

$$Re^{i\phi} = k_2 e^{i(\phi_2 - \theta_2)}$$

$$R = k_2 , \phi = \phi_2 - \theta_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1) and (2), we have

$$\phi_1 - \theta_1 = \phi_2 - \theta_2$$

$$\theta_2 - \theta_1 = \phi_2 - \phi_1$$

i.e., Angle between c_1 and c_2 is equal to c_1^1 and c_2^1 in both magnitude and orientation

$\therefore w = f(z)$ **is Conformal**

The function $f(z) = \bar{z}$ is not a conformal map because it preserves only the angles between two smooth curves but not orientation. This type of transformation is known as isogonal mapping.

Critical Points

If $f(z_0)$ is analytic at $z_0 \in D$ and $f'(z_0) = 0$, then the point z_0 is known as critical point of $f(z)$. Because critical points are zeroes of analytic function f', they are isolated.

Critical points of $z^5 - 5z$ are $z = \pm 1. \ \pm i$

Example 1

Find the images in the w-plane corresponding to the straight lines x=a, x=b, y=c and y=d under the transformation $w=z^2$

Consider

$$W = z^2$$
$$\Rightarrow \frac{dw}{dz} = 2z \neq 0$$

∴ The transformation is conformal for all z≠0

Let $z = x + iy \text{ and } w = u + iv \text{ then}$

$$u + iv = (x + iy)^2$$
$$\Rightarrow u + iv = x^2 - y^2 + i2xy$$
$$\Rightarrow u + iv = (x^2 - y^2) + i2xy$$
$$\Rightarrow u = x^2 - y^2 \quad and \ v = 2xy \text{ ----------------- } (1)$$

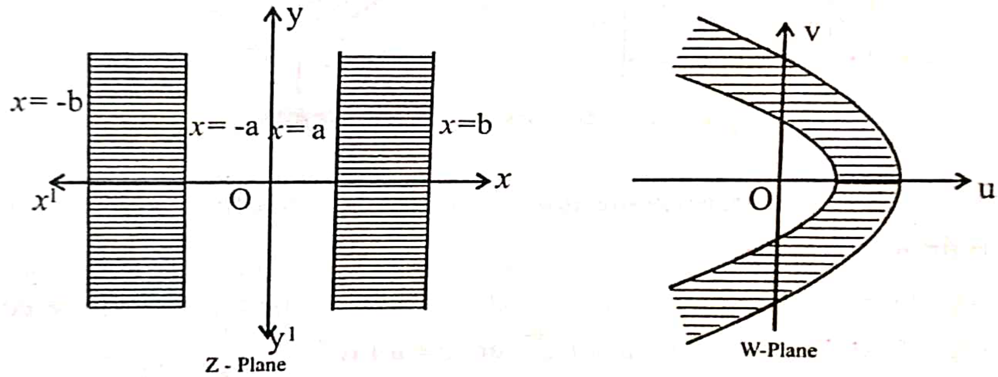
i) When x=a, (i.e., we consider the lines parallel to imaginary axis in z-plane)

Put x=a in equation (1), we get

$$u = a^2 - y^2 \text{ and } v = 2ay \Rightarrow y = \frac{v}{2a}$$
$$\therefore u = a^2 - \left(\frac{v}{2a}\right)^2$$
$$\Rightarrow 4a^2u = 4a^4 - v^2$$
$$\Rightarrow v^2 = 4a^4 - 4a^2u$$
$$\Rightarrow v^2 = 4a^2(a^2 - u) \text{ or}$$
$$v^2 = -4a^2(u - a^2)$$

Which is parabola in the w-plane with vertex at $(a^2,0)$ and symmetrical about the real axis.

Thus, the transformation $w=z^2$ transforms lines parallel to imaginary axis into a set of confocal parabolas in w-plane.



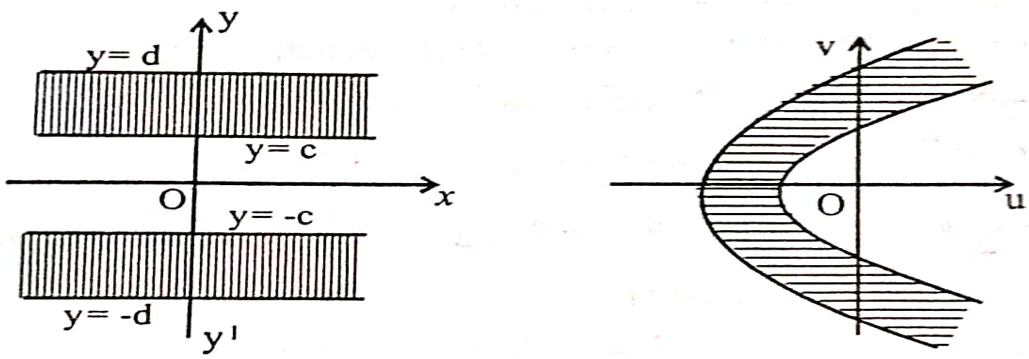
ii) When y=c (i.e., we consider the lines parallel to real axis in z-plane):

Put $y = c$ in equation (1), we get

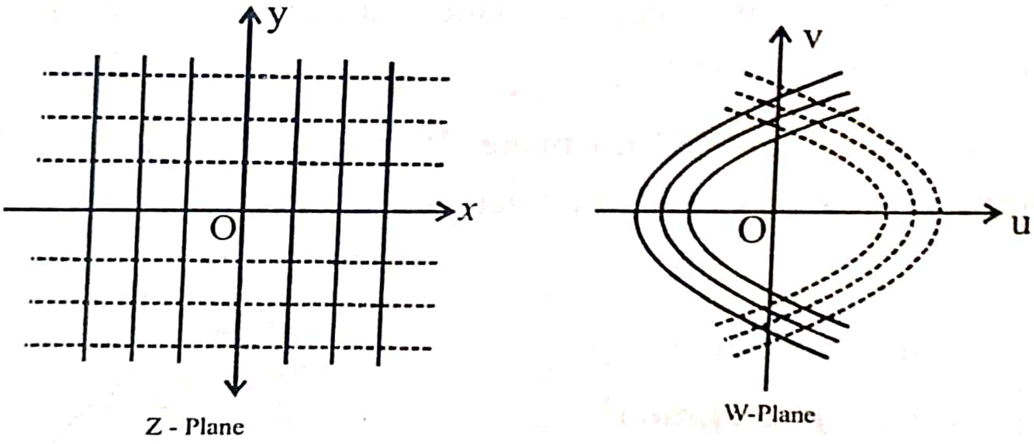
$$u = x^2 - c^2 \text{ and } v = 2xc \Rightarrow x = \frac{v}{2c}$$
$$\therefore u = \frac{v^2}{4c^2} - c^2$$
$$4c^2u = v^2 - 4c^4$$
$$v^2 = 4c^2u + 4c^4$$
$$v^2 = 4c^2(u + c^2)$$
$$v^2 = 4c^2[u - (-c^2)]$$

Which is a parabola in w-plane with vertex at $(-c^2,0)$ and symmetrical about the real axis.

Thus, the transformation $w=z^2$ transforms lines parallel to real axis into set of confocal parabolas in w-plane.



$\therefore$ The two families of parabolas $v^2 = -4a^2(u - a^2)$ and $v^2 = 4c^2(u + c^2)$ corresponds to two orthogonal families of straight lines $x = \pm a$ and $y = \pm c$



Example 2: The transformation $w = e^z$

Consider

$$W = e^z$$
$$\Rightarrow \frac{dw}{dz} = e^z \neq 0, \forall z$$
$$\therefore \text{the transformation is conformal for all } z.$$
$$\text{Let } z = x + iy \text{ and } w = u + iv \text{ then}$$
$$u + iv = e^{x+iy}$$
$$u + iv = e^x . e^{iy}$$
$$u + iv = e^x(\cos y + i \sin y)$$
$$u = e^x \cos y \text{ and } v = e^x \sin y \text{ ---------- (1)}$$

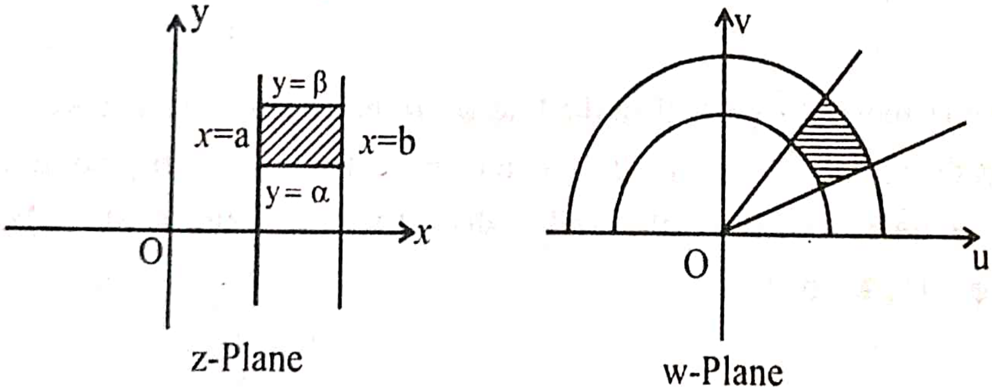
i) When x=a

Put $x = a$ in equation (1) we get

$$u = e^a \cos y \text{ and } v = e^a \sin y$$
$$\Rightarrow \frac{u}{e^a} = \cos y \text{ and } \frac{v}{e^a} = \sin y$$
$$\Rightarrow \left(\frac{u}{e^a}\right)^2 + \left(\frac{v}{e^a}\right)^2 = \cos^2 y + \sin^2 y$$
$$\Rightarrow \frac{u^2}{e^{2a}} + \frac{v^2}{e^{2a}} = 1$$
$$\Rightarrow u^2 + v^2 = e^{2a}$$

Which is a circle with center at origin and radius e^a in w-plane.

Thus, the transformation w=e^z transform the lines parallel to imaginary axis in z- plane into set of concentric circles in w-plane.



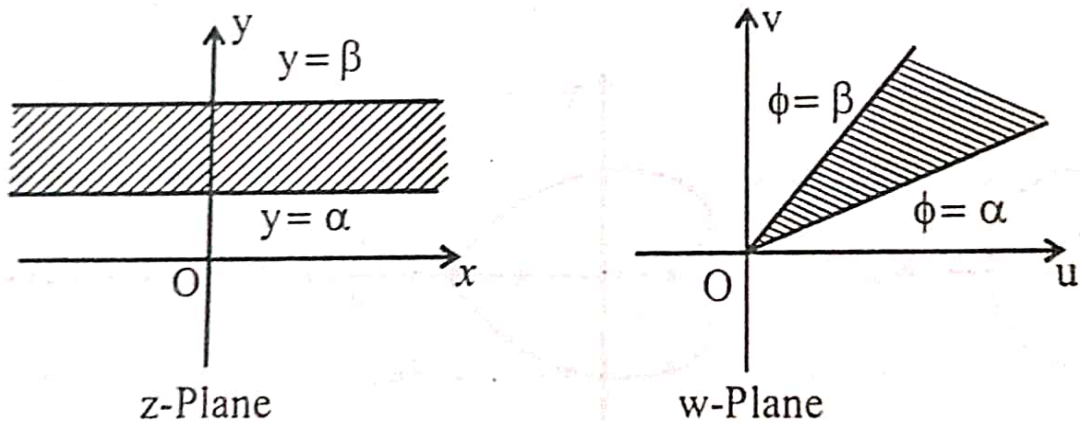
ii) When y=c

Put y=c in equation (1) we get

$$u = e^x \cos c \ \ and \ v = e^x \sin c$$
$$\Rightarrow \frac{u}{\cos c} = e^x \ and \ \frac{v}{\sin c} = e^x$$
$$\Rightarrow \ v = (\tan c)\, u$$

Which is a line passing through the origin in w-plane.

Thus, the transformation w=e^z transformation the lines parallel to real axis in z-plane into set of lines passing through the origin in w-plane.



Example 3: The transformation w=sin (z)

Consider,

W=sin (z)

$$\Rightarrow \frac{dw}{dz} = \cos(z) \neq 0 \ \ \forall\, z \neq \frac{n\pi}{2} \ \ where\, n\, is\, an\ \ odd\ integer.$$

∴ The transformation is conformal for $z \neq \frac{n\pi}{2}$ where n is an odd integer.

Let z=x+iy and w=u+iv then

$$u + iv = \sin(x + iy)$$
$$\Rightarrow \ \ u + iv = \sin x \cos(iy) + cosx \sin(iy)$$
$$\Rightarrow \ u + iv = sinxcoshy + cosx(isinhy)$$
$$\Rightarrow u + iv = sinxcoshy + icosxsinhy$$
$$\therefore \ u = sinxcoshy \ and \ v = cosxsinhy \text{ ---------- (1)}$$

i) When x=a

Put x=a in equation (1) we get

$$u = sinacoshy \ and \ v = \cos a \ sinhy$$
$$\Rightarrow \frac{u}{\sin a} = \cos hy \ and \ \frac{v}{\cos a} = \sin hy$$
$$\Rightarrow \frac{u^2}{sin^2 a} - \frac{v^2}{cos^2 a} = cosh^2 y - sinh^2 y$$
$$\Rightarrow \frac{u^2}{sin^2 a} - \frac{v^2}{cos^2 a} = 1$$

Which represents a hyperbola in w-plane.

Thus, the transformation w=sin (z) transforms lines parallel to imaginary axis into a set of hyperbola in w-plane.

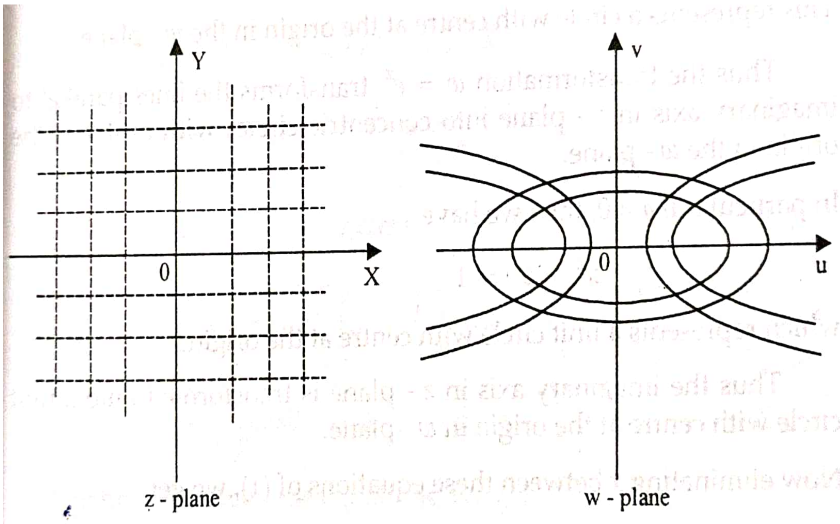
ii) When y=c

Put $y = c$ in equation (1), we get

$$u = sinxcoshc \ and \ v = \cos x \sin hc$$
$$\Rightarrow \frac{u}{\cos hc} = \sin x \text{ and } \frac{v}{\sin hc} = cosx$$
$$\Rightarrow \frac{u^2}{cosh^2 c} + \frac{v^2}{sinh^2 c} = 1$$

Which represents an ellipse in w-plane.

Thus, the transformation w=sin (z) transforms lines parallel to real axis into set of ellipses in w-plane.



Example 4: The transformation $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$

Consider, $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$$\frac{dw}{d\ z} = \frac{1}{2}\left(1 - \frac{1}{z^2}\right) = \frac{1}{2}\left(\frac{z^2-1}{z^2}\right)$$
$$\frac{dw}{dz} = 0 \quad if\ z = \pm 1$$

Thus, the transformation is conformal for all $z \neq \pm 1$

Let $w = u + iv \ and \ z = re^{i\theta}$ then

$$u + iv = \frac{1}{2}\left(re^{i\theta} + \frac{1}{re^{i\theta}}\right)$$
$$\Rightarrow u + iv = \frac{1}{2}\left[r(cos\theta + isin\theta) + \frac{1}{r}(cos\theta - isin\theta)\right]$$
$$\Rightarrow u + iv = \frac{1}{2}\left[\left(r + \frac{1}{r}\right)cos\theta + i\left(r - \frac{1}{r}\right)sin\theta\right]$$
$$\therefore u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta \ and \ v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta - - - - - (1)$$

i) When $r = a$ where a is a non-zero constant (≠1)

Put r=a in equation (1) we get

$$u = \frac{1}{2}\left(a + \frac{1}{a}\right)\cos\theta \ and \ v = \frac{1}{2}\left(a - \frac{1}{a}\right)\sin\theta$$

$$\Rightarrow \frac{u}{\frac{1}{2}\left(a+\frac{1}{a}\right)} = \cos\theta \quad and \quad \frac{v}{\frac{1}{2}\left(a-\frac{1}{a}\right)} = \sin\theta$$

$$\Rightarrow \frac{u^2}{\frac{1}{4}\left(a+\frac{1}{a}\right)^2} + \frac{v^2}{\frac{1}{4}\left(a-\frac{1}{a}\right)^2} = cos^2\theta + sin^2\theta = 1$$

Which represents an ellipse in w-plane.

Thus, the transformation, $w = \frac{1}{2}\left(z+\frac{1}{z}\right)$ transforms the concentric circles with centre at origin in z- plane into set of confocal ellipses in w- plane.

In particular if r=1 then we have $u = \cos\theta \;\; and \; v = 0$.

i.e., the unit circle |z|=1 in z-plane is mapped onto the segment of u-axis from -1 to 1 in w-plane.

ii) When θ=b where b is a constant

Put θ=b in equation (1) we get

$$u = \frac{1}{2}\left(r+\frac{1}{r}\right)\cos b \; and \; v = \frac{1}{2}\left(r-\frac{1}{r}\right)\sin b$$

$$\Rightarrow \frac{u}{\cos b} = \frac{1}{2}\left(r+\frac{1}{r}\right) \; and \; \frac{v}{\sin b} = \frac{1}{2}\left(r-\frac{1}{r}\right)$$

$$\Rightarrow \frac{u^2}{cos^2 b} - \frac{v^2}{sin^2 b} = \frac{1}{4}\left[\left(r+\frac{1}{r}\right)^2 - \left(r-\frac{1}{r}\right)^2\right]$$

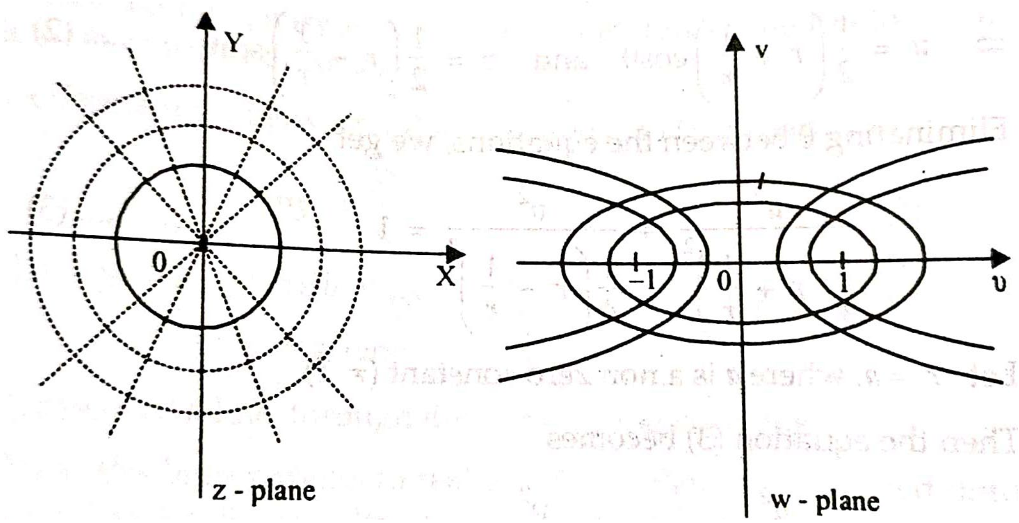
$$= \frac{1}{4}\left[r^2 + \frac{1}{r^2} + 2 - r^2 - \frac{1}{r^2} + 2\right]$$

$$= \frac{1}{4}(4)$$

$$\frac{u^2}{cos^2 b} - \frac{v^2}{sin^2 b} = 1$$

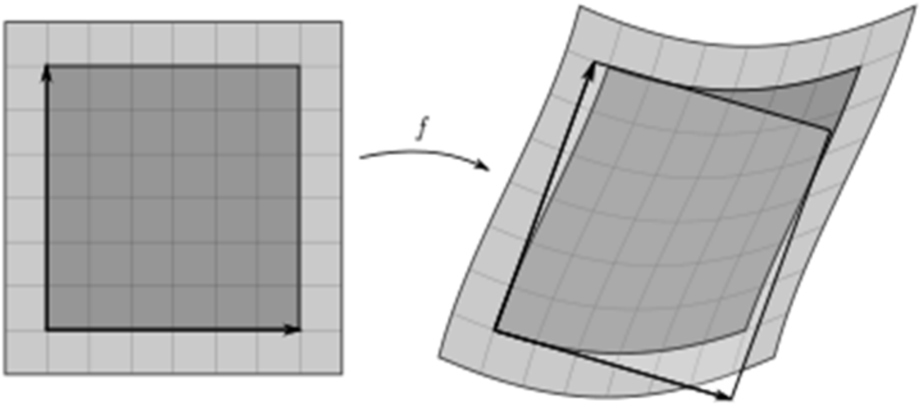
Which represents a hyperbola in w-plane.

Thus, the transformation $w = \frac{1}{2}\left(z+\frac{1}{z}\right)$ transforms the lines passing through the origin in z-plane into the set of confocal hyperbolae in w-plane.



Jacobian Transformation

The Jacobian of the transformation of the function $f(z) = u(x, y) + i\, v(x, y)$ is defined by $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ and denoted by $J_f(u, v)$.



If f(z) is analytic function then $J_f(u, v) = |u_x^2 + v_x^2| = |f'(z)|^2$.

Application

The below discussed mentioned are the important applications of Conformal Mapping.

- Conformal mapping is an important technique used in complex analysis and used in complex analysis and has many applications in different physical situations.
- Conformal mappings are used in Engineering and physics that can be expressed in terms of functions of a conformal variable but that exhibit inconvenient geometrics.
- Laplace equation can also be solved with the help of conformal mapping. We can also determine the conduction of heat by it.
- Conformal mapping has also come across advantages in fluid dynamics.
- Conformal mapping can be used in scattering and diffraction problems.

Conclusion

When we study the topic Conformal Mapping and Its Applications, there are many forms of conformal mapping which are considered as significant tools in many disciplines like physics. Engineering and mathematics. In addition, they carry wide range of application in many other fields**.**

References

1. Ahlfors L. Complex Analysis, McGraw Hill, 1979.
2. A Text Book of Mathematics, Gururajachar GB.
3. A Text Book of Mathematics, Ranganath GK.
4. Website.