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On the Order-Reversing Partial one-to-one Transformation Semigroup (IOR_n)

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Abstract
In this paper, we study the subsemigroup of all order-reversing partial one to one transformation (IOR_n). Let X_n={1,2,3,..,n} and Let α:Domα⊆X_n⟶Im⊆X_n be a partial one-to-one transformation on X_n. The elements of partial one to one transformation semigroup were constructed and a subsemigroup of order-reversing was identified. The following parameters are defined: the fix point of α, $f(\alpha) = \{x\alpha = x\}$, the height of α, $h(\alpha) = |im\alpha|$, the positive waist of α, $w^{+}(\alpha) = \max(im\alpha)$, the derangement of $\alpha, d_n(\alpha) = \{\alpha(x) \neq x\}$, the idempotent of α, $\alpha^2 = \alpha$ and cardinality of order-reversing subsemigroup $|IOR_n|$ was computed. The combinatorial results for three variable functions for the order-reversing subsemigroup discovered was enumerated using the parameters defined above.

Keywords: Cardinality, fix point, height, right waist, derangement and idempotent

1. Introduction

Let $X_n = (1,2,3, \ldots, n)$ and let $Dom\alpha \subseteq X_n$ and $Im\alpha \subseteq X_n$, then the transformation $\alpha: Dom\alpha \longrightarrow Im\alpha$ is said to be total or full if $Dom\alpha = X_n$ and strictly partial otherwise.

The height of α is denoted and defined by $h(\alpha) = |Im\alpha|$, the breadth of α is denoted and defined by $b(\alpha) = |Dom(\alpha)|$, the right waist of α is denoted and defined by $w^{+} = \max(Im\alpha)$, the left waist of α is defined and denoted by $w^{-} = \min(\alpha)$. The fix point of α (fix of α) is defined and denoted by $f(\alpha) = |F(\alpha)| = |\{x \in X_n : x\alpha = x\}|$ and idempotent of α is defined by $\alpha^2 = \alpha$ if and only if $Im\alpha = F(\alpha)$ (Garba, 1990, 1994b; Laradji and Umar, 2006, 2007, Ganyusahkin and Manzochuk 2003, Umar, 1997, 2010). The derangement of α is defined and denoted by $d_n(\alpha) = \{\alpha(x) \neq x\}$ (Bashir, 2008). The main object of study in this paper is the order-reversing partial one-to-one transformation (IOR_n). The main objectives of this paper are to compute up to three variable functions of (IOR_n) and find their integer sequence from the on-line encyclopedia sequence (Sloane, 2011).

2. IOR_n

Umar (2010) defined all order-reversing partial one-to-one transformation semigroup that for any $\alpha \in I_n$ and $x, y \in Dom\alpha: x \leq y \Longrightarrow x\alpha \geq y\alpha$.

We investigated the elements of IOR_n for $n = \{0,1,2,3,4,5\}$ and find the fix point, height, right waist, idempotent and derangement together with the cardinality.

Example 1: The semigroup IOR_2 contains the following six elements (Umar 2010).

$$\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\ 1 & \emptyset\end{pmatrix}, \begin{pmatrix}1 & 2\\ \emptyset & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\ \emptyset & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\ 2 & \emptyset\end{pmatrix}, \begin{pmatrix}1 & 2\\ \emptyset & \emptyset\end{pmatrix}.$$

2.1 Combinatorial Results for Order-Reversing (IOR_n)

First note that it seems reasonable to define $k = 0\, if\, p = 0; and\, F(n;k) = F(n;p,k) = 1$ if $k = p = 0$ this and other observations we record in the following lemma, proposition and corollaries which will be use implicitly whenever needed (Umar, 2010).

Lemma. Let $X_n = \{1,2,3,\ldots,n\} and\, P = \{p,m,k,q\}, where\, for\, a\, given\, \alpha \in IODR_n,$ we set $p = h(\alpha), m = f(\alpha), k = w^+(\alpha) and\, q = d_n(\alpha)$. We also defined $F(n;k) = F(n;p,k) = 1\, if\, k = p = 0.$

Then,
1. $n \geq k \geq p \geq m \geq 0;$
2. $k = 1 \Rightarrow p = 1;$
3. $p = 0 \Leftrightarrow k = 0.$

The following are easy to prove, but nevertheless, we include its proof to demonstrate the technique.

Theorem: Let $I_n = IOR_n$, then $|IOR_n| = \binom{2n}{n}, \ n \geq 0.$

For all the elements $\alpha = \begin{pmatrix}1 & 2 & \ldots n\\ x_1 x_2 & & \ldots n\end{pmatrix} \in IOR_n$ can be uniquely determined by $x_1, x_2, \ldots, x_n$ for $i = 1,2,3,\ldots,n.$ Set $y_i = x_i + i$, then the mapping between $(x_1, x_2, \ldots, x_n) \to (y_1, y_2, \ldots, y_n)$ is a bijection between the set of all $(x_1, x_2, \ldots, x_n)$ such that $1 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq n$ and the set of all $(y_1, y_2, \ldots, y_n)$ such that $1 \leq y_1 \leq y_2 \leq \cdots \leq y_n \leq n + n$. If follows that $(y_1, y_2, \ldots, y_n)$ is uniquely determined by the $n - elements$ subset $(y_1, y_2, \ldots, y_n)$ of $\{1,2,3,\ldots,2n\}$. Hence $\binom{2n}{n}$.

Corollary 1 Let $I_n = IOR_n$ then,

$$f(n;m) = \begin{cases} \binom{n+i-1}{mi} m! & i \geq 0, \quad m = 1, \quad 1 \leq n \leq 4 \\ (m-n) & n \geq 2, \quad m \geq 2 \\ \frac{\{1-3n+n^2-\sqrt{(1-6n+7n^2-2n^2+n^4)}\}}{2n} & n \geq 1 \ ---A078482 \end{cases}$$

Corollary 2 Let $I_n = IOR_n$ then,

$$f(n;p) = \begin{cases} \binom{n+p}{n} & n \geq 0, \quad p = 0 \\ \binom{n}{p} & n \geq 0, \quad p \geq 0 \\ \binom{n^2}{p} = n^2 p & n \geq 1, \quad p = 1 \\ \binom{n^2}{p-i} & n \geq 1, \quad p \geq 1, \quad i \geq 0 \\ \left(\frac{(n-1)((n-1)+1)}{p}\right)^2 & n \geq 2, \quad p = 2 \\ \left(\frac{(n-1)((n-1)+1)}{n-p}\right) & n \geq 2, \quad p \geq 0 \end{cases}$$

Corollary 3 Let $I_n = IOR_n$ then,

$$f(n;k) = \begin{cases} \binom{n+k-1}{k} & n \geq 1, \quad k \geq 0 \\ n+k-1 & n\geq1, \quad k=1 \\ \binom{n+k}{n} & n\geq0, \quad k=0 \\ \binom{n+1}{k} & n \geq 2, \quad k = 2 \\ \binom{2k+1}{n-1} & n \geq 2, \quad k \geq 0 \\ \binom{2k-1}{n} & n \geq 1, \quad k \geq 1 \end{cases}$$

Corollary 4 Let $I_n = IOR_n$ then,

$$f(n;\dot{a}) = \begin{cases} \dfrac{7(3^n) + 2n + 5}{4} & n \geq 2 \\ a_n = (n - \dot{a})^2 & n = \dot{a} \geq 2 \end{cases}$$

Corollary 5 Let $I_n = IOR_n$ then,

$$f(n;\dot{b}) = \begin{cases} \binom{n+1}{p+k} = \dfrac{n(n+1)}{p+k} & n \geq 1, \ p = k = 1 \\ a_n = a_{n-1} + a_{n-3} + a_{n-4} & a_0 = 1, a_1 = 2, a_3 = 3 \\ 4^n + n & n \geq 2 \\ a_n = 4a_{n-1} + a_{n-2} & a_0 = 2, a_1 = 5 \end{cases}$$

Corollary 6 Let $I_n = IOR_n$ then,

$$f(n;\dot{c}) = \{a_n = 4(3^{n-3}) \qquad a_1 = 1, a_2 = 2, \quad n \geq 3$$

Corollary 7 Let $I_n = IOR_n$ then,

$$f(n;\dot{\imath}) = \begin{cases} a_n = 2a_{n-1} + (n-1) & a_0 = 1, a_2 = 1, \quad n \geq 2 \\ a_n = a_{n-1} + a_{n-2} + a_{n-3} + 4n - 8 & n \geq 3 \end{cases}$$

Corollary 8 Let $I_n = IOR_n$ then,

$$f(n;\dot{\jmath}) = \{a_n = 2a_{n-1} + a_{n-2} - a_{n-3} \qquad a_0 = 1, a_1 = 3, a_2 = 6, \quad n \geq 3$$

Corollary 9 Let $I_n = IOR_n$ then,

$$f(n;\dot{l}) = \left\{ \dfrac{a_n = 2^n(n^3 - 3n^2 + 2n + 48}{48} \right. \qquad n \geq 0$$

Corollary 10. Let $I_n = IOR_n$ then, $f(n,q) = \{a_n = (1 + a_{n-1})\left(\frac{a_{n-2}}{a_{n-3}}\right)$ $\qquad a_0 = a_1 = a_2 = 1$

Corollary 11. Let $I_n = IOR_n$ then, the $E(IOR_n) = n + 1$ $\qquad n \geq 0$

3. Concluding Remarks

Remark 1: We have considered the order-reversing partial one-one transformation, but there still others parameters that was not considered. Umar 2010 considered the union of order-reversing and order-preserving.

Remark 2: There are many sequences of numbers as at the time of writing this paper that are not yet listed/registered in the Sloane's Encyclopedia of Integer Sequence.

Remark 3: We considered only one class of transformation subsemigroup, however, there are other classes of transformation subsemigroups that can be identified if studied.

Remark 4: We have considered only three variable functions, however one can compute forth variable functions and so on, but at the moment it seems to be a difficult proposition.

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