Volume: 2 Issue: 5 Pages: 37-41

Some Combinatorial Problems in Triad Subsemigroups of Symmetric Inverse Semigroup (I_n)

*[1]Michael Cornelius, [2]MI Bello, [3]NH Manjak and [4]Ishiaku Zubairu

*[1,4]Tutor I, Department of Mathematics Unit, School of Basic and Remedial Studies, Gombe State University, Gombe, Nigeria.

[2,3]Professor, Department of Mathematical Sciences, Abubakar Tafawa Balewa University, Bauchi, Nigeria.

Article Info.
E-ISSN: 2583-6528
Impact Factor (SJIF): 5.231
Available online:
www.alladvancejournal.com

Received: 16/April/2023
Accepted: 31/May/2023

***Corresponding Author**
Michael Cornelius
Tutor I, Mathematics Unit, School of Basic and Remedial Studies, Gombe State University, Gombe, Nigeria.

Abstract
Let X_n={1,2,3,..,n} and Let α:Domα⊆X_n⟶Im⊆X_n be a partial one-to-one transformation on X_n. The elements of partial one to one transformation semigroup were constructed and a Triad subsemigroup was identified using the intersections of order-preserving, order-reversing and order-decreasing partial one to one transformation subsemigroup. The following parameters are defined: the fix point of α, $f(\alpha)=\{x\alpha=x\}$, the height of $\alpha, \mathrm{h}(\alpha)=|im\alpha|$, the positive waist of α, $w^+(\alpha)=\max(im\alpha)$, the derangement of $\alpha, d_n(\alpha)=\{\alpha(x)\neq x\}$, the idempotent of α, $\alpha^2=\alpha$ and cardinality of the Triad subsemigroup $|IODR_n|$ of course other parameters have been defined and many more could be still be defined, but we shall restrict ourselves to only these, in this paper. The combinatorial results for three variable functions for the subsemigroup (Triad) discovered was enumerated using the parameters defined. The results were presented and highlighted open problems.

Keywords: Fix point, height, waist, derangement, idempotent and Cardinality

1. Introduction

Let $X_n=(1,2,3,\ldots,n)$ and let $Dom\alpha\subseteq X_n$ and $Im\alpha\subseteq X_n$, then the transformation $\alpha: Dom\alpha\longrightarrow Im\alpha$ is said to be total or full if $Dom\alpha=X_n$ and strictly partial otherwise.

Let denote T_n to be the set of all total or full transformation on $n-element$, P_n to be the set of all partial transformation on $n-element$ and I_n to be the set of all partial one-one transformation on $n-element$, then: the set T_n contains n^n elements and P_n contains $(n+1)^n$ elements. To see this we observe that, each element $\alpha\in T_n$ is uniquely defined by $\alpha=\begin{pmatrix}1 & 2 & \cdot & \cdot & \cdot & n\\ p_1 & p_2 & \cdot & \cdot & \cdot & p_n\end{pmatrix}$, where each $p_i\ \in N$. Since the choices of $p_i s$ are independent, we have $|T_n|=n^n$, by product rule. In case of P_n, the elements p_i can be independently chosen from the set $N\cup\{\emptyset\}$. Hence, the product rule implies$|P_n|=(n+1)^n$. and I_n contains $\sum_{p=0}^{n}\binom{n}{p}^2 p!$ elements, that is each partial injection $\alpha: N\rightarrow N$ can be considered as a bijection $\alpha: Dom(\alpha)\rightarrow Im(\alpha)$. Let us count the number of such bijections of $rank\ k$. The set $A=Dom(\alpha)$ can be chosen in $\binom{n}{p}$ different ways, the set $B=Im(\alpha)$ can be independently chosen in $\binom{n}{p}$ different ways. If $A\ and\ B$ are fixed, then there are exactly $p!$ different bijections from $A\ and\ B$. Hence we have exactly $\binom{n}{p}\cdot\binom{n}{p}\cdot p!$ bijections of $rank\ p$. since p can be an arbitrary integer between $0\, and\, n$, the statement of the theorem is obtained by applying the sum rule. (Ganyusahkin and Manzochuk, 2003; Laradji and Umar, 2006; Umar, 2010). For the purpose of illustrations, the semigroup P_n, can be expressed as $f(n;r) = \binom{n}{r}n^r, if\ n \geq r \geq 0$ then the

$\sum_{r=0}^{n} f(n;r) = F(n,0) + F(n,1) + F(n,2) + F(n,3) + \cdots + F(n,n) = \binom{n}{0} + \binom{n}{1}n + \binom{n}{2}n^2 + \cdots + \binom{n}{n}n^n = (n+1)^n$,

by (Zubairu, M. M., Bashir Ali 2018). The height of α is denoted and defined by $h(\alpha) = |Im\alpha|$, the breadth of α is denoted and defined by $b(\alpha) = |Dom(\alpha)|$, the right waist of α is denoted and defined by $w^+ = \max(Im\alpha)$, the left waist of α is defined and denoted by $w^- = \min(\alpha)$. The fix point of α (fix of α) is defined and denoted by $f(\alpha) = |F(\alpha)| = |\{x \in X_n : x\alpha = x\}|$ and idempotent of α is defined by $\alpha^2 = \alpha$ if and only if $Im\alpha = F(\alpha)$ (Garba, 1990, 1994b; Laradji and Umar, 2006, 2007; Umar, 1997, 2010). The derangement of α is defined and denoted by $d_n(\alpha) = \{\alpha(x) \neq x\}$ (Bashir, 2008). It is well-known that a partial transformation α is nilpotent if and only if $\alpha^k = \emptyset$ (the empty or zero map) for some positive integer k (Garba, 1994a,c, Umar 1997, 2010). The main object of study in this paper is the triad subsemigroup of order-preserving and order-decreasing and order-reversing partial one-to-one transformation (that is, the intersections of order-preserving, order-decreasing, and order-reversing ($IODR_n$)). The main objectives of this paper are to compute the variable functions of the triad subsemigroup ($IODR_n$) and find their integer sequence from the on-line encyclopedia sequence (Sloane, 2011).

2. Triad Subsemigroup (IODRN)

Umar, (1992a,b, 1997, 1998, 2010) defined a transformation $\alpha \epsilon I_n$ is said to be order-preserving if $\forall\, x, y\ \in Dom\alpha : x \leq y \Rightarrow x\alpha \leq y\alpha$, and order-decreasing if $\forall\, x, y\ \in Dom\alpha : x\alpha \leq x$, similarly, a transformation $\alpha \in I_n$ is said to be order-reversing if for all $x, y\ \in Dom\alpha : x\ \leq y \Rightarrow x\alpha \geq y\alpha$. Hence a subsemigroup is said to be Triad if and only if $|im\alpha| \leq 1$.

2.1 Illustrate

We now Illustrate this by Giving Example

Example 1: The subsemigroup $IODR_5$ contains the following sixteen elements (Borwein *et al.*, 1989; Gayusahkin and Manzochuk, 2003, 2009; Laradji and Umar, 2004b,c, 2007).

$$\begin{pmatrix}1&2&3&4&5\\1&\emptyset&\emptyset&\emptyset&\emptyset\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\\emptyset&1&\emptyset&\emptyset&\emptyset\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\\emptyset&\emptyset&1&\emptyset&\emptyset\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\\emptyset&\emptyset&\emptyset&1&\emptyset\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\\emptyset&\emptyset&\emptyset&\emptyset&1\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\\emptyset&2&\emptyset&\emptyset&\emptyset\end{pmatrix},$$

$$\begin{pmatrix}1&2&3&4&5\\\emptyset&\emptyset&2&\emptyset&\emptyset\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\\emptyset&\emptyset&\emptyset&2&\emptyset\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\\emptyset&\emptyset&\emptyset&\emptyset&2\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\\emptyset&\emptyset&3&\emptyset&\emptyset\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\\emptyset&\emptyset&\emptyset&3&\emptyset\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\\emptyset&\emptyset&\emptyset&\emptyset&3\end{pmatrix},$$

$$\begin{pmatrix}1&2&3&4&5\\\emptyset&\emptyset&\emptyset&4&\emptyset\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\\emptyset&\emptyset&\emptyset&\emptyset&4\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\\emptyset&\emptyset&\emptyset&\emptyset&5\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\\emptyset&\emptyset&\emptyset&\emptyset&\emptyset\end{pmatrix}$$

Example 2: The subsemigroup $IODR_5$ contains the following six-idempotent elements (Clifford and Preston, 1961; Fernandes *et al.*, 2011; Garba, 1990, 1994b; Howie, 1971, 1995).

$$\begin{pmatrix}1&2&3&4&5\\1&\emptyset&\emptyset&\emptyset&\emptyset\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\\emptyset&2&\emptyset&\emptyset&\emptyset\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\\emptyset&\emptyset&3&\emptyset&\emptyset\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\\emptyset&\emptyset&\emptyset&4&\emptyset\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\\emptyset&\emptyset&\emptyset&\emptyset&5\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\\emptyset&\emptyset&\emptyset&\emptyset&\emptyset\end{pmatrix}.$$

2.2 Combinatorial Results for $IODR_n$

First note that it seems reasonable to define $k = 0\ if\ p = 0; and\ F(n;k) = F(n;p,k) = 1$ if $k = p = 0$ this and other observations we record in the following lemma, proposition and corollaries which will be use implicitly whenever needed (Umar, 2010).

Lemma. Let $X_n = \{1,2,3,\ldots,n\} and\ P = \{p, m, k, q\}, where\ for\ a\ given\ \alpha \in IODR_n$, we set $p = \mathrm{h}(\alpha), m = f(\alpha), k = w^+(\alpha) and\ q = d_n(\alpha)$. We also defined $F(n;k) = F(n;p,k) = 1\ if\ k = p = 0$.

Then,

1. $n \geq k \geq p \geq m \geq 0;$
2. $k = 1 \Rightarrow p = 1;$
3. $p = 0 \Leftrightarrow k = 0.$

The following theorems, propositions, and corollaries are easy to prove, but nevertheless, we include its proof to demonstrate the technique.

Theorem 1: The intersection of order-preserving and order-reversing and order-decreasing partial one to one subsemigroup, is a subsemigroup with $|im\alpha| \leq 1$.

Proof.

Let $\alpha \in I_n$, and let $x \in dom\alpha$ then certainly $\alpha(x) \in im\alpha$.

Also suppose $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq \in dom\alpha$ then $\alpha(x_1) \leq \alpha(x_2) \leq \alpha(x_3) \leq \cdots \leq \alpha(x_n) \in im\alpha$. Now, α is order-preserving, if for any $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \Rightarrow \alpha(x_1) \leq \alpha(x_2) \leq \alpha(x_3) \leq \alpha(x_n)$, and order-reversing if $\alpha(x_1) \geq \alpha(x_2) \geq \alpha(x_3) \geq \cdots \geq \alpha(x_n) \forall x_n \in dom\alpha, \alpha(x_n) \in im\alpha$. Obliviously, for α to be both order-preserving and order-reversing it must have $|im\alpha| \leq 1$.

Suppose also the $|im\alpha| > 1$, then for any $\alpha = \begin{pmatrix} 1 & 2 & \cdot & \cdot & \cdot & q & \cdot & \cdot & \cdot & r & \cdot & \cdot & \cdot & n \\ \cdot & \cdot & \cdot & \cdot & \cdot & p & \cdot & \cdot & \cdot & t & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ with $t > p$, if α is order-preserving, then for $q \leq r \Rightarrow \alpha(p) \leq \alpha(t)$ and order-reversing if $q \leq t \Rightarrow \alpha(p) \geq \alpha(t)$, implies $\alpha(p) = \alpha(t)$ as $p = t$.

Therefore, $|im\alpha| = 1$, now if $\alpha(p) = \emptyset \; and \; \alpha(t) = \emptyset \Rightarrow \alpha(p) = \alpha(t) = \emptyset$ then if follows from the definition α is order-decreasing if $\alpha(x) \leq x$ which implies $x\alpha = x$. Hence $|im\alpha| \leq 1$.

Theorem 2: Let $I_n = IODR_n$ then, $|E(IODR_n)| = n + 1 \; for \; all \; n \geq 0$

Proof.

Each element $\alpha \in IODR_n$ is uniquely defined by $\alpha = \begin{pmatrix} 1 & 2 & \ldots & n \\ x_1 & x_2 & \ldots & x_n \end{pmatrix}$, where each of $x_i \in X_n$. Since, the choices of chosen the idempotent elements in $x_i s$ are all dependent on $f(\alpha) = Im(\alpha)$, except the one which is independently chosen from $X_n \cup \{\emptyset\}$, if follows that $|E(IODR_n)| = n + 1$.

Proposition. Let $I_n = IODR_n$ then the $|IODR_n| = \frac{n}{2}(n + 1) + 1 \; for \; all \; n \geq 0$.

Proof.

From the definition of $IODR_n$, it follows that for an element $\alpha \in IODR_n$ the image $\alpha(x)$ for some $x \in X_n$ can be chosen in the product of $\frac{n}{2} \; and \; (n + 1)$ different ways. However, the images of the different elements from X_n are always dependently plus one inpendently chosen from $X_n \cup \{\emptyset\}$. Hence $|IODR_n| = \frac{n}{2}(n + 1) + 1$.

The following corollaries were deduced as follows

Corollary 1. Let $I_n = IODR_n$ then, $f(n; m) = \begin{cases} \frac{n}{2}(n + 1)m! & n \geq 0, m = 0 \\ n(m!) & n \geq 0, m = 0 \\ n - m & n \geq m \geq 2 \end{cases}$

Corollary 2. Let $I_n = IODR_n$ then, $f(n; p) = \begin{cases} \binom{n}{p} & p = 0, n \geq 0 \\ \frac{n}{2}(n + p) & n \geq 0, p = 1 \\ n - p & n \geq p \geq 2 \end{cases}$

Corollary 3. Let $I_n = IODR_n$ then, $f(n; k) = \begin{cases} (n - (k - 1)) & n \geq k \geq 0 \\ (n + 1) - k & n \geq 1, k = 1 \\ (n + 1) - k & n \geq 2, k = 2 \end{cases}$

Corollary 4. Let $I_n = IODR_n$ then,

$$f(n;m,p)=\begin{cases} 2n-\binom{m}{p} & 1\le n\le 4, p=m=1 \\ f(n;m,k) & n\ge 2, p=m=0 \\ \dfrac{n!}{(p-1)!\,(|n-p-1|!)}-p & n=p=m\ge 2 \end{cases}$$

Corollary 5. Let $I_n = IODR_n$ then,

$$f(n;p,k)=\begin{cases} (kp)\wedge\left|\dfrac{3n-2}{2}\right| & 1\le n\le 3, k=p=1 \\ (k-p)\binom{n}{p-2}+\binom{k+1}{p+1}m & n=k=p\ge 0 \\ n+1-n & n\ge 0, k=p=0 \\ (n-(p-1)) & n\ge p\ge k\ge 0 \end{cases}$$

Where $\wedge$ is the join and $\vee$ is the meet.

Corollary 6. Let $I_n = IODR_n$ then,

$$f(n;k,m)=\begin{cases} (n-1)m!+\wedge\left(\dfrac{1}{4}(n^2-n+1)k!\right) & n\ge 2, m=k=0 \\ \dfrac{n}{k!}\binom{m}{k} & n\ge 1, m=k=1 \\ m-k+1 & n=m=k\ge 0 \\ (n-(m-1)) & n\ge m\ge k\ge 0 \end{cases}$$

Corollary 7. Let $I_n = IODR_n$ then,

$$f(n;m,p,k)=\begin{cases} 2(m+k)-3np & m=k=p=1, 1\le n\le 3 \\ (n-1)(m-k)p+1 & n=m=k=p\ge 0 \\ n-(m!+k!) & n\ge 3, m=k=p=0 \end{cases}$$

Corollary 8. Let $I_n = IODR_n$ then,

$$f(n;p,k,m)=\begin{cases} (n+1)-1+k_i & n\ge 1, \quad k_i=k_{i+1}, \quad i\ge 0 \\ f(n;p,m) & n\ge 2, p=k=m=0 \\ (n+p)!+\displaystyle\sum_{p=0}^{m}\binom{k+1}{m+1} & n=p=k=m=0 \end{cases}$$

Corollary 9. Let $I_n = IODR_n$ then,

$$f(n;k,m,p)=\begin{cases} (n-1)+\wedge\left|\dfrac{1}{n}(n^2-n+k!)\right| & n\ge 1, \quad k=m=p=0 \\ f(n;p,k,m) & n\ge 3, k=m=p=0 \\ f(n;m,k) & n=k=m=p\ge 0 \end{cases}$$

Corollary 10. Let $I_n = IODR_n$ then,

$$f(n;q)=n-q \qquad \forall\ (n=q\ge 0), (n\ge 1, q\ge 0), (n\ge 1, q=0)\ and\ (n\ge 3, q=1)$$

Corollary 11. Let $I_n = IODR_n$ then, $|E(IODR_n)| = n+1$ $\qquad n\ge 0$

3. Remarks

Remark 1: All these combinatorial functions can be computed when restricted to special subsets or sets within a particular transformation. For example, the set of idempotent Garba GU (1990) and Howie JM (1971). The set of Nilpotent, Garba GU (1994c) and Laradji A. and Umar A. (2004).

Remark 2: We have considered only one class of transformation subsemigroup, Umar have considered seven classes of the transformation semigroups, Umar A (2010). However, there are other classes of transformation semigroups that can be identified if studied.

Remark 3: The study of semigroups helps us to understand the events in our lives by understanding the order in which they occur. e.g. sequences and series play an important role in various aspects of lives, they are used in predicting, evaluating, monitoring and decision making of the outcome of situation or events.

Remark 4: The combinatorial functions always give rise to sequences of numbers and triangular of numbers which are fascinating/pivotal tool in machine language e.g. Automata. Hence the need to study these combinatorial problems from this point.

Remark 5: Ideally, we would like to compute forth variable functions and so on for any finite semigroup of all partial one to one transformations, but at the moment this seems to be a difficult proportion and so we have to start from the smaller variable function to higher variable functions.

Remark 6: The main statement proved above is by direct combinatorial arguments, however, this approach does not always work and finding a recurrences and guessing a closed formula which can then be proved by mathematical induction is another approach that can be effectively use. Umar A. et al are currently using generating functions to investigate some of the unknown cases and it looks very promising.

References

1. Bashir Ali, Umar A. Some Combinatorial Properties of the Alternating Group. Southeast Asian Bulletin of Mathematics, 2008.
2. Borwein D, Rankin, Renner L. Enumeration of Injective Partial Transformation. Discrete Math. 1989; 73:291-296.
3. Clifford A, Preston G, The Algebraic Theory of Semigroups. Vol.1, Providence, R.I, American Mathematical Society, 1961.
4. Fernandes V, Gomes GMS, Jesus M. The Cardinal and Idempotent Number of Various Monoid of Transformation on a Finite Chain. Bull Malays. Math. Sci. Soc. 2011; 34:79-85.
5. Ganyusahkin E, Manzochuk V. On the Structure of IO_n. Semigroup Forum. 2003; 66:455-483.
6. Ganyusahkin E, Manzochuk V. Classical Finite Transformation Semigroup: An Introduction, Springer, London, 2009.
7. Garba GU. Idempotents in Partial Transformation Semigroup. Portugal Mathematica. 1990; 51:163-172.
8. Garba GU On the Nilpotent Ranks of Partial Transformation Semigroups. Potugal Mathematica. 1994a; 51:185-204.
9. Garba GU. On the Idempotent Ranks of Certain Semigroups of Preserving Mapping. Semigroup Forum. 1994b; 51:185-204.
10. Garba GU. Nilpotent in Semigroups of Partial One-to-one Order-preserving Mapping. Semigroup Forum. 1994c; 48:37-49.
11. Howie JM. Products of Idempotents in Certain Semigroup of Transformations. Proc. Edinburgh Math. Soc. 1971; 17:223-236.
12. Howie JM. Fundamentals of Semigroup Theory. Oxford: Clarendon Press, 1995.
13. Laradji A, Umar A. On Number of Nilpotents in the Partial Symmetric Semigroup. Comm. Algebra 2004; 32:3017-3023.
14. Laradji A, Umar A. Combinatorial Results for Semigroups of Order-preserving Partial Transformations. J. Algebra. 2004b; 278:342-359.
15. Laradji A, Umar A. Combinatorial Results for Semigroups of Order-decreasing Partial Transformations. J. Integer Seq. 2004c; 7:04.3.8.
16. Laradji A, Umar A. Combinatorial Results for Semigroups of Order-preserving Full Transformation. Semigroup Forum. 2006; 72:51-62.
17. Laradji A, Umar A. Combinatorial Results for the Symmetric Inverse Semigroup. Semigroup Forum. 2007; 75:221-236.
18. Laradji A, Umar A. Combinatorial results for semigroups of order-preserving or order-reversing subperrmutations (in preparation)
19. Liu CL. Introduction to Combinatorial Results Mathematics. McGraw Hill Company, New York, 1968.
20. Stanley RP. Enumerative Combinatorics. Cambridge University Press, 1997, I.
21. Stanley RP. Enumerative Combinatorics. Second Edition, 2011, I.
22. Sloane NJA. (Ed.) The On-line Encyclopedia of Integer Sequences, 2011. Available at http://oeis.org
23. Umar A. Semigroups of Order-decreasing transformation, Ph.D. Thesis, University of St. Andrews, 1992a.
24. Umar A. On the Semigroups of Order-decreasing Finite Full Transformations. Proc. Roy. Soc. Edinburgh. 1992b; 120:129-142.
25. Umar A. On the Infinite Semigroups of Order-decreasing Full Transformations. I, Comm. Algebra. 1997; 25:2987-2999.
26. Umar A. Enumeration of Certain Finite Semigroups of Transformations. Discrete Math. 1998; 89:291-297.
27. Umar A. Some Combinatorial Problems in the Theory of Symmetric Inverse Semigroups. Discrete Math. 2010; 9:115-126.
28. Zubairu MM, Bashir Ali. On Certain Combinatorial Problems of the Semigroup partial and Full Contraction of a Finite Chain. Bayero Journal of pure and applied Science. 2018; 11(1):377-380.